MATH AND MUSIC

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1. The Circle of Fifths

Sound is the result of a vibrating object in air. The number of vibrations in a given second is called the frequency of the tone the object produces. Certain ratios of frequencies occur naturally in the Western music system and produce more harmonious sounds. Some of these natural ratios are given in the table below:

Table 1. Common Intervals

Interval	Ratio
Octave	2:1
Perfect Fifth	3:2
Perfect Fourth	4:3
Whole Step	9:8

Legend has attributed these ratios to Pythagoras who heard the sounds of the hammers of four smiths. These hammers had weights 12, 9, 8, and 6 pounds. Upon investigating which hammers sounded pleasant when hit together, he derived the intervals above. In the following we will focus on the perfect fifth.

Fix a frequency (wlog say this frequency is 1) and denote the frequency by the letter C. We will construct a complete scale from this frequency by applying the two operations and their inverse given below.

- Doubling the frequency moves up an octave.
- Multiplying the frequency by $\frac{3}{2}$ moves up a perfect fifth.

By applying these rules on a standard piano we arrive at the progression of notes in Table 2. Note that on the standard piano we loop back to our starting point. However there is a problem. The final note we arrive at is at the ratio $\frac{531441}{262144} = \frac{3^{12}}{2^{18}}$. However since our progression has gone up an octave from the starting point we would expect the frequency to be 2. In fact the ratio of the frequency we arrived at to 2 is

$$\frac{531441}{262144 \cdot 2} = 1.013643264770...$$

TABLE 2. The Circle of Fifths

Note	Frequency
C	1
G	3/2
D	9/8
A	27/16
E	81/64
B	243/128
F$\sharp$	729/512
C$\sharp$	2187/2048
G$\sharp$	6561/4096
D$\sharp$	19683/16384
A$\sharp$	59049/32768
E$\sharp$	177147/131072
B$\sharp$	531441/262144

instead of 1. This discrepancy between Pythagoras' notion of harmonic intervals and our standars piano with a 12 note scale is known as the *Pythagorean Comma*.

2. A Fatal Flaw

What went wrong here? Essentially the problem boils down to the fact that twelve perfect fifths do not result in seven octaves (recall we had to scale by $\frac{1}{2}$ twelve times in our construction). Mathematically

$$\left(\frac{3}{2}\right)^1 2 = 129.746337891... \neq 128 = 2^7.$$

or

$$2^{19} \neq 3^{12}.$$

Being mathematicians we can generalize this problem. We are looking for positive integers k, y such that if we produce notes by using perfect fifths, after y notes we will have moved by k octaves. This results in solving the equation

$$\left(\frac{3}{2}\right)^y = 2^k.$$

Or just solving the equation

$$3^y = 2^x$$

for some positive integers x and y. However from a quick application of the unique factorization theorem, we see that this can not be done.

An inherent discrepancy between the Pythagorean harmony and our twelve note scale.

Yet all is not lost. In a sense, as we will show, the twelve note scale is actually as good as we can do.

3. Continued Fractions

Time for some actual mathematics. After taking logarithms we can rewrite the above equation as

$$\frac{x}{y} = \log_2 3.$$

Of course this can not be solved for integers x, y since $\log_2 3$ is irrational. A quick computation shows

$$\log_2 3 = 1.584962500721156181...$$

Note that this decimal expansion produces a sequence of rational approximations to $\log_2 3$. Can we do better then this? Our choice of rational approximation here was fairly arbitrary. In fact our choice of base 10 representation was fairly arbitrary. A more natural way to think about numbers is through continued fractions.

A continued fraction is an expansion of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

where $a_1, a_2, ...$ are a sequence of positive integers and $a_0 \in \mathbb{Z}$. Represent this continued fraction in the form $[a_0, a_1, a_2, ...]$ and let $\frac{p_n}{q_n} = [a_0, a_1, a_2, ..., a_n]$ be the rational n^{th} convergent of our continued fraction. The value of a continued fraction is defined to be the limit of its convergents.

In particular every number has a continued fraction expansion, and if the number is rational this expansion terminates.

To find the continued fraction expansion of a number A begin by computing $a_0 = \lfloor A \rfloor$. Now $A = a_0 + x_0$ so $0 \leq x_0 \leq 1$. Now let $a_1 = \lfloor \frac{1}{x_0} \rfloor$ and then $\frac{1}{x_0} = a_1 + x_1$ for some x_1. Repeat this process. If A is rational eventually the $x_n = 0$. So

$$A = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}.$$

In a sense continued fractions are the most natural representations for a number as the following theorem demonstrates:

Theorem 3.1. *Let x be an irrational number, $n \geq 1$, and let p_n/q_n with $\gcd(p_n, q_n) = 1$ be the n^{th} convergent of the continued fraction of x. Let $p, q \in \mathbb{Z}$ with $0 \leq q \leq q_n$ and $p/q \neq p_n/q_n$, then*

$$\left| x - \frac{p_n}{q_n} \right| < \left| x - \frac{p}{q} \right|.$$

To understand this result consider the following question: "What makes a rational number a good approximation to a given real number?" Since the rationals are dense in the reals, we can produce infinitely many approximations to a given real number. So it is not good enough for a rational to be close to a real number. A more precise measure is given by the size of the denominator of a rational number. So a good rational approximation depends on how close we can approximate a real number with a small denominator rational number.

Thus in a sense the theorem tells us that the n^{th} convergent gives the best rational approximation to x among all rationals whose denominator is no greater then q_n.

4. The 12 Note Scale

Let us return to the musical problem of finding solutions to

$$\frac{x}{y} = \log_2 3.$$

Since we can not actually find solutions we will approximate $\frac{x}{y}$ by the convergents to $\log_2 3$.

The continued fraction expansion of $\log_2 3$ is $[1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, ...]$. The first few convergents are

$$1, 2, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \frac{84}{53}.$$

First consider the fifth approximation,

$$\frac{x}{y} \approx \frac{19}{12}$$

and

$$3 = 2^{\log_2 3} \approx 2^{19/12}$$

or

$$\frac{3}{2} \approx 2^{7/12}.$$

Now recall that y represented the number of notes in our scale. By using this approximation we see that we can use twelve notes with a fifth corresponding to $2^{7/12}$. So an octave should be divided into twelve equal intervals and a fifth corresponds to some seven of them.

Amazingly western music has adopted the fourth best approximation to Pythagorean Harmony using equal temperament.

Note that we could use another approximation to develop a different note scale. For example Chinese Music uses a five note scale and people have developed forty one note scales. The following gives a summary of these scales.

12 note scale

- The perfect fifth is $2^{7/12} \approx 1.498307...$

5 note scale

- The perfect fifth is $2^{3/5} \approx 1.51571657...$

41 note scale

- The perfect fifth is $2^{24/41} \approx 1.50041943...$

References:

Dunne, Edward, McConnell, Mark, *Pianos and Continued Fractions.* Mathematics Magazine, Vol. 72, No. 2. (Apr., 1999), pp. 104-115.