Waterloo Math Circles - Combinatorics

1 Factorials

Say we wish to order 20 different books in a row, how many ways can we accomplish this? Well, there are 20 possibilities for the first book, 19 for the second book, 18 for the third book etc. Thus the total number of possibilities is just

$$20 \cdot 19 \cdot 18 \cdot 17 \cdots 1 = 20!.$$

In general we can arrange n items in $n!$ ways.

The factorial of a number also has some interesting number-theoretic properties that often appear on contests, try the following problems.

Problem 1. How many 0's are at the end of 50!?

Problem 2. Find all positive integers n such that $1! + 2! + \cdots n!$ is the square of a positive integer.

Problem 3. Determine n such that $n! = (2^{15})(3^6)(5^3)(7^2)(11)(13)$.

2 Basic Counting

How many ways can we remove three books from the twenty books we have in such a way that the order we remove matters? Well there are 20 possibilites for the first book, 19 for the second and 18 for the third book. Thus there are a total of $20 \cdot 19 \cdot 18$ possibilities. In general if we need to choose k objects from n objects in an ordered fashion, we can do this in $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$ ways.

How about if we don't care about the order in which the books are? Well in this case we can pull out $20 \cdot 19 \cdot 18$ ordered sets of 3 books, but to remove the ordering we can divide by 3!, so the number of ways of choosing three books from 20 is $(20 \cdot 19 \cdot 18)/(3 \cdot 2 \cdot 1)$.

Define $\binom{n}{k} = \frac{n!}{k!(n-k)!}$., This gives the number of ways to choose k objects from n objects.

Problem 4. Prove this.

Problem 5. Two students must be chosen out of a group of thirty for a math contest, how many ways can this be done?

Problem 6. How many ways can we arrange the letters in MATHEMATICS? How about MISSISSIPPI?

Problem 7. Determine the number of integers n which satisfy all three of the following:

- Each digit of n is 0 or 1.
- n is divisible by 6.
- $0 < n < 10^7$

Problem 8. How many ways can you toss a coin 10 times so that 2 or more succesive heads will appear at least once. (Hint: Can you count the number of ways it won't occur?)

3 Lattice Paths

How many paths are there from $(0, 0)$ to (n, k), with n and k positive, which consist of upward and rightward moves only? Notice that any path can be understood as a string of U's (up's) and R's (rights). Thus we are interested in enumerating all strings of length $n + k$ with k U's or n R's. This can be done in $\binom{n+k}{k}$ ways. (Example on board)

Problem 9. CMO 2005 Question 1

Problem 10. (Hard) How many paths are there from $(0, 0)$ to (n, k), with $n \leq k$, which do not cross the line $y = x$. They are allowed to touch the line. (Hint: Count the number of paths to (n, k) and subtract the paths which cross the diagonal. Can you say something about the paths which cross? What happens if you rotate your original path? What happens if you rotate part of your original path?)